AUTOMORPHISMS OF CENTRAL EXTENSIONS OF TYPE I VON NEUMANN ALGEBRAS

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Abstract. Given a von Neumann algebra M we consider the central extension $E(M)$ of M. For type I von Neumann algebras $E(M)$ coincides with the algebra $LS(M)$ of all locally measurable operators affiliated with M. In this case we show that an arbitrary automorphism T of $E(M)$ can be decomposed as $T = T_a \circ T_\phi$, where $T_a(x) = axa^{-1}$ is an inner automorphism implemented by an element $a \in E(M)$, and T_ϕ is a special automorphism generated by an automorphism ϕ of the center of $E(M)$. In particular if M is of type I_∞ then every band preserving automorphism of $E(M)$ is inner.

1. Introduction

In the series of paper [1]-[3] we have considered derivations on the algebra $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra M, and on various subalgebras of $LS(M)$. A complete description of derivations has been obtained in the case of von Neumann algebras of type I and III.

A comprehensive survey of recent results concerning derivations on various algebras of unbounded operators affiliated with von Neumann algebras is presented in [4].

It is well-known that properties of derivations on algebras are strongly correlated with properties of automorphisms of underlying algebras (see e.g. [8]). Algebraic automorphisms of C^*-algebras and von Neumann algebras were considered in the paper of R. Kadison and J. Ringrose [9], which is devoted to automatic continuity and innerness of automorphisms. By this paper we initiate a study of automorphisms of the algebra $LS(M)$ and its various subalgebras. In the commutative case a similar problem has been considered by A.G. Kusraev [12] who proved by means of Booolean-valued analysis the existence of non trivial band preserving automorphism on algebras of the form $L^0(\Omega, \Sigma, \mu)$. The algebra $LS(M)$ and its subalgebras present a non commutative counterparts of the algebra $L^0(\Omega, \Sigma, \mu)$. In the present paper we establish a general form of automorphisms of the algebra $LS(M)$ for type I von Neumann algebras M.

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Let $\mathcal{A}$ be an algebra. A one-to-one linear operator $T : \mathcal{A} \to \mathcal{A}$ is called an *automorphism* if $T(xy) = T(x)T(y)$ for all $x, y \in \mathcal{A}$. Given an invertible element $a \in A$ one can define an automorphism T_a of $\mathcal{A}$ by $T_a(x) = axa^{-1}$, $x \in \mathcal{A}$. Such automorphisms are called *inner automorphisms* of $\mathcal{A}$. It is clear that for commutative (abelian) algebra $\mathcal{A}$ all inner automorphisms are trivial, i.e. acts as unit operator. In the general case inner automorphisms are identical on the center of $\mathcal{A}$. Essentially different classes of automorphisms are those which are generated by automorphisms of the center $Z(\mathcal{A})$ of $\mathcal{A}$. In some cases such automorphisms ϕ on $Z(\mathcal{A})$ can be extended to automorphisms T_ϕ of the whole algebra $\mathcal{A}$ (see e.g. Kaplansky [10, Theorem 1]). The main result of the present paper shows that for a type I von Neumann algebra M every automorphism T of the algebra $LS(M)$ can be uniquely decomposed as a composition $T = T_a \circ T_\phi$ of an inner automorphism T_a and an automorphism T_ϕ generated by an automorphism ϕ of the center of $LS(M)$.

In section 2 we recall the notions of the algebras $S(M)$ of measurable operators and $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra M. We also introduce the so-called *central extension* $E(M)$ of the von Neumann algebra M. In the general case $E(M)$ is a *-subalgebra of $LS(M)$, which coincides with $LS(M)$ if and only if M does not have direct summands of type II. We also introduce two generalizations of the topology of convergence locally in measure on $LS(M)$ and prove that for the type I case they coincide.

In section 3 we consider automorphisms of the algebra $E(M)$ – the central extension of a von Neumann algebra M. We prove (Theorem 3.10) that if M is of the type I then each automorphism T of $E(M)$ which acts identically on the center $Z(E(M))$ of $E(M)$, is inner. We also show that for homogeneous type I von Neumann algebras M every automorphism ϕ of the center $Z(E(M))$ of $E(M)$ can be extended to an automorphism T_ϕ of the whole $E(M)$. Finally we prove the main result of the present paper which shows that each automorphism T of $E(M)$ for a type I von Neumann algebra M can be uniquely represented as $T = T_a \circ T_\phi$, where T_a is an inner automorphism implemented by an element $a \in E(M)$, and T_ϕ is an automorphism generated by an automorphism ϕ of the center of $E(M)$. In particular we obtain that each bundle preserving automorphism of $E(M)$ is inner if M is of type I_∞.

2. Central extensions of von Neumann algebras

In this section we give some necessary definitions and a preliminary information concerning algebras of measurable and locally measurable operators affiliated with a von Neumann algebra. We also introduce the notion of the central extension of a von Neumann algebra.

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Consider a von Neumann algebra M in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M.

A linear subspace $\mathcal{D}$ in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra M, if

1) $\mathcal{D}\eta M$;

2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in M.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. Denote by $S(M)$ the set of all measurable operators with respect to M (see [14]).

A closed linear operator x in H is said to be *locally measurable* with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$ (see [15]).

It is well-known [5], [15] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator, and contains $S(M)$ as a solid *-subalgebra.

Let (Ω, Σ, μ) be a measure space and from now on suppose that the measure μ has the direct sum property, i. e. there is a family $\{\Omega_i\}_{i\in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup\limits_{i\in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0(\Omega, \Sigma, \mu)$ the algebra of all (equivalence classes of) complex measurable functions on (Ω, Σ, μ) equipped with the topology of convergence in measure.

Consider the algebra $S(Z(M))$ of operators which are measurable with respect to the center $Z(M)$ of the von Neumann algebra M. Since $Z(M)$ is an abelian von Neumann algebra it is *-isomorphic to $L^\infty(\Omega, \Sigma, \mu)$ for an appropriate measure space (Ω, Σ, μ). Therefore the algebra $S(Z(M))$ coincides with $Z(LS(M))$ and can be identified with the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable functions on (Ω, Σ, μ).

The basis of neighborhoods of zero in the topology of convergence locally in measure on $L^0(\Omega,\Sigma,\mu)$ consists of the sets

$$W(A,\varepsilon,\delta)=\{f\in L^0(\Omega,\Sigma,\mu):\exists B\in\Sigma,\ B\subseteq A,\ \mu(A\setminus B)\leq\delta,$$

$$f\cdot\chi_B\in L^\infty(\Omega,\Sigma,\mu),\ ||f\cdot\chi_B||_{L^\infty(\Omega,\Sigma,\mu)}\leq\varepsilon\},$$

where $\varepsilon,\delta>0,\ A\in\Sigma,\ \mu(A)<+\infty$, and χ_B is the characteric function of the set $B\in\Sigma$.

Recall the definition of the dimension functions on the lattice $P(M)$ of projection from M (see [5], [14]).

By L_+ we denote the set of all measurable functions $f:(\Omega,\Sigma,\mu)\to[0,\infty]$ (modulo functions equal to zero μ-almost everywhere).

Let M be an arbitrary von Neumann algebra with the center $Z=L^\infty(\Omega,\Sigma,\mu)$. Then there exists a map $D:P(M)\to L_+$ with the following properties:

(i) $d(e)$ is a finite function if only if the projection e is finite;

(ii) $d(e+q)=d(e)+d(q)$ for $p,q\in P(M),\ eq=0$;

(iii) $d(uu^*)=d(u^*u)$ for every partial isometry $u\in M$;

(iv) $d(ze)=zd(e)$ for all $z\in P(Z(M)),\ e\in P(M)$;

(v) if $\{e_\alpha\}_{\alpha\in J},\ \ e\in P(M)$ and $e_\alpha\uparrow e$, then

$$d(e)=\sup_{\alpha\in J}d(e_\alpha).$$

This map $d:P(M)\to L_+$, is a called the *dimension functions* on $P(M)$.

Remark 2.1. Recall that for an element $x\in M$ the projection defined as

$$c(x)=\inf\{z\in P(Z(M)):zx=x\}$$

is called the central cover of x.

Let M be a type I von Neumann algebra. If $p,q\in P(M)$ are abelian projections with $c(p)=c(q)=\mathbf{1}$, then the property (iii) implies that $0<d(p)(\omega)=d(q)(\omega)<\infty$ for μ-almost every $\omega\in\Omega$. Therefore replacing d by $d(p)^{-1}d$ we can assume that $d(p)=c(p)$ for every abelian projection $p\in P(M)$. Thus for all $e\in P(M)$ we have that $d(e)\geq c(e)$.

The basis of neighborhoods of zero in the topology $t(M)$ of *convergence locally in measure* on $LS(M)$ consists (in the above notations) of the following sets

$$V(A,\varepsilon,\delta)=\{x\in LS(M):\exists p\in P(M),\ \exists z\in P(Z(M)),\ xp\in M,$$

$$||xp||_M\leq\varepsilon,\ \ z^\perp\in W(A,\varepsilon,\delta),\ \ d(zp^\perp)\leq\varepsilon z\},$$

where $\varepsilon,\delta>0,\ A\in\Sigma,\ \mu(A)<+\infty$.

The topology $t(M)$ is metrizable if and only if the center $Z(M)$ is σ-finite (see [5]).

Given an arbitrary family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in M with $\bigvee\limits_{i\in I}z_i=\mathbf{1}$ and a family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a

unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum\limits_{i \in I} z_i x_i$.

We denote by $E(M)$ the set of all elements x from $LS(M)$ for which there exists a sequence of mutually orthogonal central projections $\{z_i\}_{i \in I}$ in M with $\bigvee\limits_{i \in I} z_i = \mathbf{1}$, such that $z_i x \in M$ for all $i \in I$, i.e.

$$E(M) = \{x \in LS(M) : \exists z_i \in P(Z(M)), z_i z_j = 0, i \neq j, \bigvee_{i \in I} z_i = \mathbf{1}, z_i x \in M, i \in I\},$$

where $Z(M)$ is the center of M.

It is known [3] that $E(M)$ is *-subalgebras in $LS(M)$ with the center $S(Z(M))$, where $S(Z(M))$ is the algebra of all measurable operators with respect to $Z(M)$, moreover, $LS(M) = E(M)$ if and only if M does not have direct summands of type II.

A similar notion (i.e. the algebra $E(\mathcal{A})$) for arbitrary *-subalgebras $\mathcal{A} \subset LS(M)$ was independently introduced recently by M.A. Muratov and V.I. Chilin [6]. The algebra $E(M)$ is called *the central extension of* M.

It is known ([3], [6]) that an element $x \in LS(M)$ belongs to $E(M)$ if and only if there exists $f \in S(Z(M))$ such that $|x| \leq f$. Therefore for each $x \in E(M)$ one can define the following vector-valued norm

$$||x|| = \inf\{f \in S(Z(M)) : |x| \leq f\} \tag{2.1}$$

and this norm satisfies the following conditions:

1)$\|x\| \geq 0; \|x\| = 0 \Longleftrightarrow x = 0$;
2)$\|fx\| = |f|\|x\|$;
3)$\|x + y\| \leq \|x\| + \|y\|$;
4)$||xy|| \leq ||x||||y||$;
5)$||xx^*|| = ||x||^2$
for all $x, y \in E(M), f \in S(Z(M))$.

Let us equip $E(M)$ with the topology which is defined by the following system of zero neighborhoods:

$$O(A, \varepsilon, \delta) = \{x \in E(M) : ||x|| \in W(A, \varepsilon, \delta)\},$$

where $\varepsilon, \delta > 0,\ \ A \in \sum,\ \ \mu(A) < +\infty$.

Denote the above topology by $t_c(M)$.

Proposition 2.2. *The topology $t_c(M)$ is stronger that the topology $t(M)$ of convergence locally in measure.*

Proof. It is sufficient to show that

$$O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta). \tag{2.2}$$

Let $x \in O(A, \varepsilon, \delta)$, i.e. $||x|| \in W(A, \varepsilon, \delta)$. Then there exists $B \in \Sigma$ such that

$$B \subseteq A,\ \ \mu(A \setminus B) \leq \delta,$$

and

$$||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu), \ \ |||\|x\|\chi_B||_M \le \varepsilon.$$

Put $z = p = \chi_B$. Then $||xp|| = ||x\chi_B|| = ||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu)$, i.e. $xp \in M$ and moreover $||xp||_M \le \varepsilon$. Since $\mu(A \setminus B) \le \delta$ and $z^\perp \chi_B = \chi_B^\perp \chi_B = 0$, one has $z^\perp \in W(A, \varepsilon, \delta)$. Therefore

$$||xp||_M \le \varepsilon, \ \ z^\perp \in W(A, \varepsilon, \delta), \ \ zp^\perp = \chi_B \chi_B^\perp = 0$$

and hence $x \in V(A, \varepsilon, \delta)$. □

Proposition 2.3. *If M is a type I von Neumann algebra and $0 < \varepsilon < 1$, then*

$$O(A, \varepsilon, \delta) = V(A, \varepsilon, \delta).$$

Proof. From above (2.2) we have that $O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta)$. Therefore it is sufficient to show that $V(A, \varepsilon, \delta) \subset O(A, \varepsilon, \delta)$.

Let $x \in V(A, \varepsilon, \delta)$. Then there exist $p \in P(M)$ and $z \in P(Z(M))$ such that

$$xp \in M, \ \ ||xp||_M \le \varepsilon, \ \ z^\perp \in W(A, \varepsilon, \delta), \ \ d(zp^\perp) \le \varepsilon z.$$

Since M is of type I Remark 2.1 implies that $d(zp^\perp) \ge c(zp^\perp)$. Now from $d(zp^\perp) \le \varepsilon z$ it follows that $c(zp^\perp) \le \varepsilon z$. From $0 < \varepsilon < 1$ we obtain that $zp^\perp = 0$. Therefore $z \le c(p)$, where $c(p)$ is the central cover of p. Thus $z = zp$. Put $z = \chi_E$ for an appropriate $E \in \Sigma$. Since $z^\perp \in W(A, \varepsilon, \delta)$ one has that $\chi_{\Omega \setminus E} \in W(A, \varepsilon, \delta)$. Thus there exists $B \in \Sigma$ such that $B \subseteq A$, $\mu(A \setminus B) \le \delta$, $|\chi_{\Omega \setminus E}\chi_B| \le \varepsilon < 1$. Hence $\chi_B \le \chi_E$. So we obtain

$$||x||\chi_B \le ||x||\chi_E = ||x||z = ||xz|| = ||xzp|| = ||xp|| \le \varepsilon.$$

This means that $x \in O(A, \varepsilon, \delta)$. □

Corollary 2.4. *If M is a type I von Neumann algebra then the topologies $t(M)$ and $t_c(M)$ coincide.*

Proposition 2.5. *Let M be a type I von Neumann algebra and $x \in LS(M)$, $x \ge 0$. If $pxp = 0$ for all abelian projections $p \in M$ then $x = 0$.*

Proof. Since $x \ge 0$ we have that $x = yy^*$ for an appropriate $y \in LS(M)$. Then

$$0 = pxp = pyy^*p = py(py)^*$$

and hence $py = 0$. Therefore $y^*py = 0$ for all abelian projections $p \in M$. But since M has the type I there exists a family $\{p_i\}_{i \in J}$ of mutually orthogonal abelian projections such that $\sum\limits_{i \in J} p_i = \mathbf{1}$. For any finite subset $F \subseteq J$ put $p_F = \sum\limits_{i \in F} p_i$. Since $p_F \uparrow \mathbf{1}$ from $yp_Fy^* = 0$ we have that $yy^* = 0$, i.e. $x = yy^* = 0$. □

3. Automorphisms of central extensions for type I von Neumann algebras

Let $\mathcal{A}$ be an arbitrary algebra with the center $Z(\mathcal{A})$ and let $T : \mathcal{A} \to \mathcal{A}$ be an automorphism. It is clear that T maps $Z(\mathcal{A})$ onto itself. Indeed for all $a \in Z(\mathcal{A})$ and $x \in \mathcal{A}$ one has

$$T(a)T(x) = T(ax) = T(xa) = T(x)T(a)$$

which means that $T(a) \in Z(\mathcal{A})$.

An operator $T : \mathcal{A} \to \mathcal{A}$ is said to be $Z(\mathcal{A})$-linear if $T(ax) = aT(x)$ for all $a \in Z(\mathcal{A})$ and $x \in \mathcal{A}$. It is easy to see that an automorphism $T : \mathcal{A} \to \mathcal{A}$ of a unital algebra $\mathcal{A}$ is $Z(\mathcal{A})$-linear if and only if it is identical on the center $Z(\mathcal{A})$.

Theorem 3.1. *Let M be a von Neumann algebra of type I and let $E(M)$ be its central extension. Then each $Z(E(M))$-linear automorphism T of the algebra $E(M)$ is inner.*

Proof. Let us show that T is $t(M)$-continuous. First suppose that the center $Z(M)$ of the von Neumann algebra M is σ-finite. Then the topology $t(M)$ is metrizable and hence it is sufficient to prove that the operator T is $t(M)$-closed.

Consider a sequence $\{x_n\} \subset E(M)$ such that $x_n \xrightarrow{t(M)} 0$, $T(x_n) \xrightarrow{t(M)} y$. Take $x \in E(M)$ such that $T(x) = y$ and let us show that $x = 0$. Since

$$x^*x_n \xrightarrow{t(M)} 0$$

and

$$T(x^*x_n) = T(x^*)T(x_n) \xrightarrow{t(M)} T(x^*)y = T(x^*)T(x) = T(x^*x),$$

we may suppose (by replacing the sequence $\{x_n\}$ by the sequence $\{x^*x_n\}$) that $x \geq 0$.

Let $p \in M$ be an arbitrary abelian projection with $c(p) = \mathbf{1}$. Then $px_np = a_np$ for an appropriate $a_n \in S(Z(M))$, $n \in \mathbb{N}$. Since $x_n \xrightarrow{t(M)} 0$ and $c(p) = \mathbf{1}$ it follows that $a_n \xrightarrow{t(M)} 0$. Therefore

$$T(p)T(x_n)T(p) = T(px_np) = T(a_np) = a_nT(p) \xrightarrow{t(M)} 0.$$

On the other hand

$$T(p)T(x_n)T(p) \xrightarrow{t(M)} T(p)yT(p),$$

thus $T(p)yT(p) = 0$ and hence

$$pxp = T^{-1}(T(p)yT(p)) = T(0) = 0,$$

i.e. $pxp = 0$ for all abelian projections with $c(p) = \mathbf{1}$. Therefore Proposition 2.5 implies that $x = 0$, i.e. T is $t(M)$-continuous.

Now consider the general case, i.e. when the center $Z(M)$ is arbitrary. Take a family $\{z_i\}_{i \in I}$ of mutually orthogonal central projections in M with $\bigvee\limits_i z_i = \mathbf{1}$

such that $z_i Z(M)$ is σ-finite for all $i \in I$. From the above we have that $z_i T$ is $t(z_i M)$ continuous on $z_i E(M)$ for all $i \in I$, where $(z_i T)(x) = T(z_i x) = z_i T(x)$ is the restriction of T onto $z_i E(M)$ which is well-defined in view of the $Z(E(M))$-linearity of T. Therefore T is $t(M)$-continuous of whole $E(M) = \bigoplus\limits_{i\in I} z_i E(M)$.

Further by Corollary 2.4 the topologies $t(M)$ and $t_c(M)$ coincide and hence T is also $t_c(M)$-continuous and according to [16, Theorem 2] there exists $c \in S(Z(M))$ such that $||T(x)|| \leq c||x||$ for all $x \in E(M)$.

Take a sequence $\{z_n\}_{n\in\mathbb{N}}$ of mutually orthogonal central projections in M with $\bigvee\limits_n z_n = \mathbf{1}$ such that $z_n c \in Z(M)$ for all $n \in \mathbb{N}$. This means that the automorphism $z_n T$ maps bounded elements from $z_n E(M)$ to bounded elements, i.e. $z_n T(z_n M) \subseteq z_n M$. Then given any $n \in \mathbb{N}$ the automorphism $z_n T|_{z_n M}$ is identical on the center of $z_n M$. By theorem of Kaplansky [11, Theorem 10] there exist elements $a_n \in z_n M$ which are invertible in $z_n M$, such that $z_n T(x) = a_n x a_n^{-1}$ for all $x \in z_n M$. Put $a = \sum\limits_{n\geq 1} z_n a_n$. It is clear that $a \in E(M)$ and

$$T(x) = \sum_{n\geq 1} z_n T(x) = \sum_{n\geq 1} z_n T(z_n x) = \sum_{n\geq 1} a_n (z_n x) a_n = axa^{-1}$$

for all $x \in E(M)$. □

Let M be a von Neumann algebra of type I_n, $n \in \mathbb{N}$, with the center $Z(M)$. Then M is *-isomorphic to the algebra $M_n(Z(M))$ of all $n\times n$ matrices over $Z(M)$ (cf. [13, Theorem 2.3.3]). Moreover the algebra $S(M) = E(M)$ is *-isomorphic to the algebra $M_n(Z(S(M)))$, where $Z(S(M)) = S(Z(M))$ is the center of $S(M)$ (see [2, Proposition 1.5]). If e_{ij}, $i, j = \overline{1, n}$ are matrix units in $M_n(S(Z(M)))$ then each element $x \in M_n(S(Z(M)))$ has the form

$$x = \sum_{i,j=1}^{n} a_{ij} e_{ij},\ a_{ij} \in S(Z(M)),\ i, j = \overline{1, n}.$$

Let $\phi : S(Z(M)) \to S(Z(M))$ be an automorphism. Setting

$$T_\phi\left(\sum_{i,j=1}^{n} a_{ij} e_{ij}\right) = \sum_{i,j=1}^{n} \phi(a_{ij}) e_{ij} \tag{3.1}$$

we obtain a linear operator T_ϕ on $M_n(S(Z(M)))$, which is in fact an automorphism of $M_n(S(Z(M)))$. Indeed, for

$$x = \sum_{i,j=1}^{n} a_{ij} e_{ij}, y = \sum_{i,j=1}^{n} b_{ij} e_{ij},\ a_{ij}, b_{ij} \in S(Z(M)),\ i, j = \overline{1, n}$$

we have

$$T_\phi(xy) = T_\phi\left(\sum_{i,j=1}^{n} a_{ij} e_{ij} \sum_{k,s=1}^{n} b_{ks} e_{ks}\right) = T_\phi\left(\sum_{i,j,s=1}^{n} a_{ij} b_{js} e_{is}\right) =$$

$$= \sum_{i,j,s=1}^{n} \phi(a_{ij}b_{js})e_{is} = \sum_{i,j,s=1}^{n} \phi(a_{ij})\phi(b_{js})e_{is} =$$
$$= \sum_{i,j=1}^{n} \phi(a_{ij})e_{ij} \sum_{k,s=1}^{n} \phi(b_{ks})e_{ks} = T_\phi(x)T_\phi(y),$$

i.e. $T_\phi(xy) = T_\phi(x)T_\phi(y)$.

The following property immediately follows from the definition of T_ϕ :

if φ and ϕ are two automorphisms of $S(Z(M))$ then $T_\phi \circ T_\varphi = T_{\phi\circ\varphi}$, in particular $T_\phi^{-1} = T_{\phi^{-1}}$.

Remark 3.2. (i) If the automorphism ϕ on $S(Z(M))$ is non trivial (i.e. not identical) then it is clear that T_ϕ can not be an inner automorphism on $M_n(S(Z(M))$.

(ii) It is known [9, Lemma 1] that every (algebraic) automorphism of C^*-algebra is automatically norm continuous. But in our case this is not true in general. Suppose that the abelian algebra $S(Z(M))$ is represented as $L^0(\Omega, \Sigma, \mu)$, with a continuous Boolean algebra Σ. Then A.G. Kusraev [12, Theorem 3.4] has proved that $S(Z(M))$ admits a non trivial band preserving automorphism which is, in particular $t(M)$-discontinuous. Therefore T_ϕ gives an example of a $t(M)$-discontinuous automorphism of $E(M)$. In particular, T_ϕ is not inner.

Proposition 3.3. *If M is a von Neumann algebra of type I_n, then each automorphism T of $E(M)$ can be uniquely represented in the form*

$$T = T_a \circ T_\phi, \tag{3.2}$$

where T_a is an inner automorphism implemented by an element $a \in E(M)$, and T_ϕ is the automorphism of the form (3.1) generated by an automorphism ϕ of the center $S(Z(M))$.

Proof. Let ϕ be the restriction of T onto the center $Z(E(M)) = S(Z(M))$. As it was mentioned earlier ϕ map $Z(E(M))$ onto itself, i.e. φ is an automorphism of $Z(E(M))$. Consider the automorphism T_ϕ defined by (3.1) and put $S = T \circ T_\phi^{-1}$. Since T and T_ϕ coincide on $Z(E(M))$, one has that S is identical on the center $Z(E(M))$, i.e. S is a $Z(E(M))$-linear automorphism of $E(M)$. By Theorem 3.1 there exists an invertible element $a \in E(M)$ such that $S = T_a$, i.e. $S(x) = axa^{-1}$ for all $x \in E(M)$. Therefore $T = S \circ T_\phi = T_a \circ T_\phi$.

Suppose that $T = T_a \circ T_\phi = T_b \circ T_\varphi$ for $a, b \in E(M)$ and automorphisms ϕ and φ of $Z(E(M))$. Then $T_b^{-1} \circ T_a = T_\varphi \circ T_\phi^{-1}$, i.e. $T_{b^{-1}a} = T_{\varphi\circ\phi^{-1}}$. Since $T_{b^{-1}a}$ is identical on the center $Z(E(M))$ of $E(M)$, it follows that $\varphi \circ \phi$ is identical on the center $Z(E(M))$, i.e. $\varphi = \phi$. Therefore $T_\varphi = T_\phi$, i.e. $T_b^{-1} \circ T_a = Id$ and hence $T_a = T_b$. ☐

Proposition 3.4. *Let M be a von Neumann algebra and let $T : E(M) \to E(M)$ be an automorphism. If $x \in E(M)$ and its central cover $c(x) = \mathbf{1}$ then $c(T(x)) = \mathbf{1}$.*

Proof. Let $c(x) = \mathbf{1}$ and consider a central projection $z \in P(Z(M))$ such that $T(z) = \mathbf{1} - c(T(x))$. Then

$$T(zx) = T(z)T(x) = (\mathbf{1} - c(T(x))c(T(x))T(x) = 0$$

and hence $zx = 0$. Therefore $zc(x) = 0$, i.e. $z = 0$. This means that $0 = T(0) = \mathbf{1} - c(T(x)) = \mathbf{1}$, i.e. $c(T(x)) = \mathbf{1}$. □

If ϕ is a *-automorphism of $E(M)$ then it is an order automorphism and hence maps M onto M. But for an arbitrary automorphism (non adjoint preserving), this not true in general. For some particular cases one can obtain a positive result.

Proposition 3.5. *Let M be an abelian von Neumann algebra and let $\phi : E(M) \to E(M)$ be a $t(M)$-continuous automorphism. Then $\phi(M) \subseteq M$.*

Proof. Let $x \in M$ be a simple element, i.e.

$$x = \sum_{i=1}^{n} \lambda_i e_i,$$

where $\lambda_i \in \mathbb{C}, e_i \in P(M), e_i e_j = 0, i \neq j, i, j = \overline{1, n}$. Let us prove that $\phi(x) \in M$ and $||\phi(x)||_M = ||x||_M$. Since M is abelian and $\phi(e_i)^2 = \phi(e_i)$, it follows that $\phi(e_i)$ is a projection for each $i = \overline{1, n}$. Therefore from the equality

$$\phi(x) = \sum_{i=1}^{n} \lambda_i \phi(e_i)$$

we obtain that $\phi(x) \in M$ and moreover

$$||\phi(x)||_M = \max_{1 \leq i \leq n} |\lambda_i| = ||x||_M.$$

Let now $x \in M$ be an arbitrary element. Consider a sequence of simple elements $\{x_n\}$ in M which $t(M)$-converges to x and $|x_n| \leq |x|$ for all $n \in \mathbb{N}$. Then $\phi(x_n) \stackrel{t(M)}{\longrightarrow} \phi(x)$ and $||\phi(x_n)||_M = ||x_n||_M \leq ||x||_M$ for all $n \in \mathbb{N}$. Therefore $|\phi(x)| \leq ||x||_M \mathbf{1}$, i.e. $\phi(x) \in M$. □

We are now in a position to consider automorphisms of central extensions for type I_∞ von Neumann algebras.

Proposition 3.6. *Let M be a von Neumann algebra of type I_∞, and let $T : E(M) \to E(M)$ be an automorphism of the central extension $E(M)$ of M. Then T is $t(Z(M))$-continuous on $E(Z(M))$ and maps $Z(M)$ onto itself.*

Proof. Since M is of type I_∞, there exists a sequence of mutually orthogonal abelian projections $\{p_n\}_{n=1}^\infty$ in M with central covers equal to $\mathbf{1}$. For a bounded sequence $\{a_n\}$ from $Z(M)$ put

$$x = \sum_{n=1}^{\infty} a_n p_n.$$

Then

$$xp_n = p_n x = a_n p_n$$

for all $n \in \mathbb{N}$.

Now let T be an automorphism of $E(M)$ and denote by ϕ its restriction onto the center of $E(M)$. If $q_n = T(p_n)$, $n \in \mathbb{N}$, then we have

$$T(xp_n) = T(x)T(p_n) = T(x)q_n$$

and

$$T(xp_n) = T(a_n p_n) = T(a_n)T(p_n) = \phi(a_n)q_n,$$

therefore

$$T(x)q_n = \phi(a_n)q_n.$$

For the center-valued norm $\|\cdot\|$ on $E(M)$ (see (2.1)) we have

$$\|q_n\|\|\|T(x)\| \geq \|q_n T(x)\| = \|\phi(a_n)q_n\| = |\phi(a_n)|\|q_n\|,$$

i.e.

$$\|q_n\|\|\|T(x)\| \geq |\phi(a_n)|\|q_n\|.$$

Since $c(q_n) = c(p_n) = \mathbf{1}$ (Proposition 3.4) the latter inequality implies that

$$||T(x)\| \geq |\phi(a_n)|. \tag{3.3}$$

Let us show that ϕ is $t(Z(M))$-continuous on $E(Z(M))$. If we suppose the opposite, then there exists a bounded sequence $\{a_n\}$ in $Z(M)$ such that $\{\phi(a_n)\}$ is not $t(Z(M))$-bounded, which contradicts (3.3). Thus ϕ is $t(Z(M))$-continuous and Proposition 3.5 implies that T maps $Z(M)$ onto itself. $\square$

Remark 3.7. The $t(Z(M))$-continuity of T on the center $E(Z(M))$ easily implies that the restriction of T on $E(Z(M))$ and hence on $Z(M)$ is a *-automorphism (cf. [9, Lemma 1]).

Now we are going to show that similar to the case of type I_n ($n \in \mathbb{N}$) von Neumann algebras, automorphisms of the algebras $E(M)$ for homogeneous type I_α von Neumann algebras (α is an infinite cardinal numbers) also can be represented in the form (3.2).

Suppose that $\phi : Z(M) \to Z(M)$ is an automorphism. According to [10, Theorem 1] ϕ can be extended to a *-automorphism of M, which we denote by T_ϕ. Since each *-automorphism is an order isomorphism and each hermitian element of $E(M)$ is an order limit of hermitian elements from M, we can naturally extend T_ϕ to a *-automorphism of $E(M)$.

Theorem 3.8. *If M is a type I_α von Neumann algebra, where α is an infinite cardinal number, then each automorphism T on $E(M)$ can be uniquely represented as*

$$T = T_a \circ T_\phi,$$

*where T_a is an inner automorphism implemented by an element $a \in E(M)$ and T_ϕ is an *-automorphism, generated by an automorphism ϕ of the center $Z(M)$ as above.*

Proof. Let M be an automorphism of $E(M)$ where M is a type I_α von Neumann algebra with the center $Z(M)$. If ϕ is the restriction of T onto the center $S(Z(M))$ of $E(M)$, then by Proposition 3.6 ϕ maps $Z(M)$ onto itself. By [10, Theorem 1] as above ϕ can be extended to a *-automorphism of $E(M)$. Now similar to the Proposition 3.3 there exists an element $a \in E(M)$ such that $T = T_a \circ T_\phi$ and this representation is unique. $\square$

Proposition 3.9. *Let M and N be von Neumann algebras of type I and suppose that M is homogeneous of type I_α. If there exists an isomorphism (not necessary *-isomorphism) T from $E(M)$ onto $E(N)$ then N is also of type I_α.*

Proof. Let z_N be a central projection in N such that $z_N N$ is of type I_β, where β is a cardinal number. Take a central projection z_M in M such that $T(z_M) = z_N$. Replacing M and N by $z_M M$ and $z_N N$ respectively we may assume that $z_M = \mathbf{1}_M$, $z_N = \mathbf{1}_N$.

Let $\{p_i\}_{i\in I}$ (respectively $\{e_j\}_{j\in j}$) be a family of mutually equivalent and orthogonal abelian projections in M (respectively in N) with $\bigvee\limits_{i\in I} p_i = \mathbf{1}_M$, (respectively $\bigvee\limits_{j\in J} e_j = \mathbf{1}_N$,) where $|I| = \alpha, |J| = \beta$. It is clear that $c(p_i) = \mathbf{1}_M$ for all $i \in I$.

Then $q_i = T(p_i)$ is an idempotent ($q_i^2 = q_i$) but not a projection in general. Let $f_i = s_l(q_i)$ be the left projection of the idempotent q_i. Since f_i is the projection onto the range of the idempotent q_i we have that $q_i f_i = f_i$, i.e. $f_i q_i f_i = f_i$, and moreover $c(f_i) = \mathbf{1}_N$, because $c(q_i) = \mathbf{1}_N$ (see Proposition 3.4). The equalities

$$q_i E(N) q_i = T(p_i E(M) p_i) = T(Z(E(M))p_i) = E(Z(N))q_i,$$

imply that for each $x \in E(N)$ there exists $a_x \in E(Z(N))$ such that $q_i x q_i = a_x q_i$.

Now we show that f_i is an abelian projection. For $x \in E(N)$ and each f_i there exist $a_i \in E(Z(N))$ such that

$$q_i f_i x f_i q_i = a_i q_i.$$

Thus

$$f_i x f_i = (f_i q_i f_i) x (f_i q_i f_i) = f_i (q_i f_i x f_i q_i) f_i = f_i a_i q_i f_i = a_i f_i q_i f_i = a_i f_i,$$

i.e. $f_i E(N) f_i = E(Z(N)) f_i$. This means that f_i is an abelian projection.

Case 1. α and β are finite. Let Φ be a normed center-valued trace on N. Then

$$\mathbf{1}_N = \Phi(\mathbf{1}_N) = \sum_{i\in I} \Phi(q_i) = \alpha\Phi(q_1) = \alpha\Phi(f_1 q_1) = \alpha\Phi(f_1 q_1 f_1) = \alpha\Phi(f_1).$$

Since N is of type I_β, we have that

$$\mathbf{1}_N = \beta\Phi(f_1).$$

Therefore $\alpha = \beta$.

Case 2. α and β are infinite. For a faithful normal semi-finite trace τ on N put

$$\tau_i(x) = \tau(f_i x), x \in N.$$

For each $i \in I$ set

$$J_i = \{j \in J : \tau_i(e_j) \neq 0.\}$$

Since $\{e_j\}$ is an orthogonal family, one has that J_i is countable for each $i \in I$.

Suppose that there exists $j \in J$ such that $\tau_i(e_j) = 0$ for all $i \in I$. Since $\tau(f_i e_j f_i) = \tau(f_i e_j) = \tau_i(e_j) = 0$, we obtain that $f_i e_j f_i = 0$. But from

$$0 = f_i e_j f_i = f_i e_j e_j f_i = f_i e_j (f_i e_j)^*$$

it follows that $f_i e_j = 0$ for all $i \in I$. And since $\bigvee_{i \in I} f_i = \mathbf{1}_N$, this implies that $e_j = 0$ – a contradiction. Therefore given any $j \in J$ there exists $i \in I$ such that $\tau_i(e_j) \neq 0$, i.e. $j \in J_i$. Hence

$$J = \bigcup_{i \in I} J_i,$$

i.e.

$$\beta \leq \alpha \aleph_0,$$

therefore $\beta \leq \alpha$. Similarly $\alpha = \beta$.

This means that every homogeneous direct summand of the von Neumann algebra N is of type I_α, i.e. N itself is homogeneous of type I_α. $\square$

It is well-known [13] that if M is an arbitrary von Neumann algebra of type I with the center $Z(M)$ then there exists an orthogonal family of central projections $\{z_\alpha\}_{\alpha \in J}$ in M with $\sup_{\alpha \in J} z_\alpha = \mathbf{1}$ such that M is $*$-isomorphic to the C^*-product of von Neumann algebras $z_\alpha M$ of type $\mathrm{I}_\alpha, \alpha \in J$, i.e.

$$M \cong \bigoplus_{\alpha \in J} z_\alpha M.$$

In this case by definition of the central extension we have that

$$E(M) = \prod_{\alpha \in J} E(z_\alpha M).$$

Suppose that T is an automorphism of $E(M)$ and ϕ is its restriction onto the center $E(Z(M))$. Let us show that T maps each $z_\alpha E(M) \cong E(z_\alpha M)$ onto itself. The automorphism T maps $z_\alpha E(M)$ onto $T(z_\alpha)E(M)$. From Proposition 3.9 it follows that the von Neumann algebra $T(z_\alpha)M$ is of type I_α. Thus $T(z_\alpha) \leq z_\alpha$. Suppose that $z'_\alpha = z_\alpha - T(z_\alpha) \neq 0$. By Proposition 3.9 we have that $T^{-1}(z'_\alpha)M$ is of type I_α, i.e.

$$0 \neq z''_\alpha = T^{-1}(z'_\alpha) \leq z_\alpha.$$

On other hand

$$T(z_\alpha z''_\alpha) = T(z_\alpha)T(z''_\alpha) = T(z_\alpha)z'_\alpha = T(z_\alpha)(z_\alpha - T(z_\alpha)) = T(z_\alpha) - T(z_\alpha) = 0,$$

i.e. $z_\alpha z''_\alpha = 0$. Therefore since $z''_\alpha \leq z_\alpha$ we have that $z''_\alpha = 0$, – a contradictions with the inequality $z''_\alpha \neq 0$. Hence $z'_\alpha = 0$, i.e. $T(z_\alpha) = z_\alpha$.

Therefore ϕ generates an automorphism ϕ_α on each $z_\alpha S(Z(M)) \cong Z(E(z_\alpha M))$, for $\alpha \in J$. Let T_{ϕ_α} be the automorphism of $z_\alpha E(M)$ generated by $\phi_\alpha, \alpha \in J$. Put

$$T_\phi\left(\{x_\alpha\}_{\alpha\in J}\right) = \{T_{\phi_\alpha}(x_\alpha)\},\ \{x_\alpha\}_{\alpha\in J} \in E(M). \tag{3.4}$$

Then T_ϕ is an automorphism of $E(M)$.

Now we can state the main result of the present paper.

Theorem 3.10. *If M is a type I von Neumann algebra, then each automorphism T of $E(M)$ can be uniquely represented in the form*

$$T = T_a \circ T_\phi,$$

where T_a is an inner automorphisms implemented by an element $a \in E(M)$ and T_ϕ is an automorphism of the form (3.4).

Proof. Let T be an automorphism of $E(M)$ and ϕ be its restriction on $Z(E(M))$ – the center of $E(M)$. Consider the automorphism T_ϕ on $E(M)$ generated by the automorphism ϕ as in (3.4) above. Similar to the proof of Proposition 3.3 we find an element $a \in E(M)$ such that $T = T_a \circ T_\phi$ and show that this representation is unique. □

Recall [7], [12] that an operator $T : E(M) \to E(M)$ is called *band preserving* if $T(zx) = zT(x)$ for all $z \in P(Z(M))$, $x \in E(M)$.

Proposition 3.6 and Theorem 3.10 imply the following result which is an analogue of [9, Theorem 5, Remark A] giving a sufficient condition for innerness of algebraic automorphisms.

Corollary 3.11. *If M is a von Neumann algebra of type I_∞ then each band preserving automorphism of $E(M)$ is inner.*

Proof. Let ϕ be the restriction of T onto $E(Z(M))$. Since T is band preserving it follows that ϕ acts identically on the simple elements from $Z(M)$. Proposition 3.6 implies that ϕ is $t(Z(M))$-continuous. Hence ϕ is identical on the whole $S(Z(M)) = E(Z(M))$ and therefore by Theorem 3.10 T is an inner automorphism. □

Remark 3.12. It is clear that the conditions of the above Corollary is also necessary for the innerness of automorphisms of $E(M)$.

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